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On generalized synchronization of different-order chaotic systems: a submanifold approach

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Abstract

Regulation theory is used to address the synchronization phenomena of chaotic systems. Our results are based on the solution of the Francis–Isidori–Byrnes equations to derive the synchronization submanifold. Thus conditions for complete or partial synchronization are depicted. This analysis is not restrictive with respect to the master and the slave systems' dimensions, therefore it can be applied to strictly different systems with the same order or even different-order systems. Finally, workbench examples are presented to illustrate the results.

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1. Introduction

Synchronization of chaotic systems is an interesting topic that, since early 1990s, has caught the attention of the nonlinear science community. Two research directions have been already embarked upon in synchronizing chaos: (i) analysis and (ii) synthesis. The analysis problem comprises (a) the classification of synchronization phenomena (Femat and Solis-Perales 1999, Brown and Kocarev 2000); (b) the comprehension of the synchronization properties as, for instance, robustness (Kocarev *et al* 2000) or geometry; and (c) the construction of a general framework for unifying chaotic synchronization (Brown and Kocarev 2000, Boccaleti *et al* 2001). On the other hand, the synthesis of synchronization focuses on the problem of finding the control effort such that two chaotic systems share the same time evolution in some sense (see, e.g., among others, Ott *et al* (1990), Cicogna and Fronzoni (1990), Loskutov (2001) and Chacón (2006). Both the analysis and synthesis directions are active research areas, and one of the current challenges is to achieve and explain the synchronization of the chaotic system with different models. In fact, the study of such systems makes sense in several systems, such as those with different fractal dimensions (Boccaletti *et al* 2000), neural levels (Xiaofeng and Lai 2000), message transmission (Femat *et al* 2001) or respiratory/circulatory coupling (Femat and Solís-Perales 2002).

In regard to the analysis of strictly different systems, the reported studies have been focused on the existence of synchronization manifolds for coupled systems. These studies have shown that such manifolds are strongly dependent on measures from Lyapunov exponents (Josic 2000, Martens *et al* 2002). Synchronization of different models has been addressed in nonidentical space-extended systems (for the case of parameter mismatching) (Boccaletti *et al* 1999) and structurally nonequivalent systems including delay (Boccaletti *et al* 2000). In Josic (2000), the chaotic synchronization has also been analysed from invariant manifolds in terms of the existence of a diffeomorphism between the attractor of coupled systems, which is closely related to generalized synchronization (GS). Josic (2000) had also included synchronization of different systems, and illustrative examples have shown the existence of synchronization manifolds (e.g. between Rössler and Lorenz). This analysis has departed from rigorous definitions and is thorough for the complete synchronization (i.e. the synchronization of all master states with all corresponding states of the slave system, Femat and Solis-Perales (1999)). Unfortunately, such a formalism for other synchronization phenomena (as, for example, the partial-state synchronization, Femat and Solis-Perales (1999)) is still obscure. Additionally, the generalized synchronization problem between different-order systems is still open, and few works have pointed in this direction; however, all efforts have been focused on particular systems (see for instance, Ge and Yang (2008), Rodríguez *et al* (2008)) and general results must be established.

The problem of analysing the synchrony of chaotic systems consists in studying states arising due to coupling, possibly diffusive. From such a premise, diverse techniques are exploited towards synchronization of different systems (Femat and Solis-Perales 2008). One of the solutions can be derived from the tracking problem in dynamical systems. In this problem the dynamical systems are subjected to external disturbances and reference forces. The external forces are represented as solutions for a dynamical system, namely the exosystem. The regulation problem can be mathematically formulated as the problem of finding an interconnection structure (named the feedback scheme) such that an equilibrium point of the system is asymptotically stable under the feedback. Roughly speaking, the tracking error approaches zero even under the influence of external forces. The regulation problem has been extensively studied in control theory for linear systems (Francis 1977). These ideas have been then extended to the nonlinear dynamical systems (Isidori and Byrnes 1990), where it is demonstrated that the corresponding solution depends on the solution of a pair of matrix partial differential equations, known henceforth as the Francis–Isidori–Byrnes equations. Although the regulation and the synchronization problems seem different and have distinct mathematical genesis, some similarities allow us to exploit the theory onto the regulation problem to explain synchronization phenomena.

In this paper, borrowing the regulation theory from the control framework, we address the synchronization problem of nonlinear systems with not necessarily the same dimension in order to explain mechanisms for the complete and partial synchronization. This paper is organized as follows. In section 2, the regulation theory is presented; then, an analogy of this theory is applied to the synchronization in section 3 to derive conditions for complete or partial-state synchronization. The analysis is carried out firstly for a general nonlinear systems and then we focus our attention on a class of dynamical systems. Workbench examples are analysed in section 4. Finally, this paper is closed with some concluding remarks.

2. Fundamental regulation theory

In the literature on the control of dynamical systems, the regulation problem is often addressed as forcing the output of a dynamical system to reach a predetermined reference signal. Although this is the case for many systems, due to their nature, for others, such as synchronization systems, varying reference signals are imposed to obtain a suitable behaviour. In this section, a brief review of results in regard to the regulation problem is presented. Let us consider the following nonlinear time-invariant system:

$$\dot{x}_S = F_S(x_S, w, u), \tag{1}$$

$$e = h(x_S, w), \tag{2}$$

where equation (1) describes the dynamics of a plant, whose state x_S is defined in a neighbourhood U of the origin in $\mathbb{R}^m$, with a control input $u \in \mathbb{R}^p$ and subject to a set of exogenous input variables $w \in \mathbb{R}^n$ which includes disturbances to be rejected and/or references to be tracked. We consider that the first approximation matrices of system (1) are respectively, $A = [\partial F_S/\partial x_S]_{(x_S,w,u)=(0,0,0)}$ and $B = [\partial F_S/\partial u]_{(x_S,w,u)=(0,0,0)}$. Equation (2) defines an error variable $e \in \mathbb{R}^p$ expressed as the function $h : U \times \mathbb{R}^n \to \mathbb{R}^p$. Let us assume that the exogenous input w is a family of all functions of time that are the solution of a homogeneous differential equation

$$\dot{w} = F_w(w), \tag{3}$$

with the initial condition $w(0)$ ranging in a neighborhood W of the origin in $\mathbb{R}^n$. System (3) is a mathematical generator of all possible external input forces and is better known as *the exosystem*. Moreover, it is assumed that F_S, h and F_w are the smooth functions and, without loss of generality, it is also assumed that $F_S(0, 0, 0) = 0$, $F_w(0) = 0$ and $h(0, 0) = 0$. Thus, for $u = 0$, the composite system (1)–(3) has an equilibrium state $(x_S, w) = (0, 0)$ yielding zero error.

Formally speaking, the *state feedback regulation problem* for system (1)–(3) is defined as tracking the reference signals and rejecting the disturbance signals while maintaining the closed-loop stability property. The regulation problem can be formulated as the problem of determining a certain submanifold of the state space (x_S, w), where the tracking error e is zero, which is rendered attractive and invariant by the feedback. Then the *nonlinear regulation problem* (NRP) consists in finding a function $u = \alpha(x_S, w)$ such that the following conditions hold:

C1. *Stability.* The equilibrium point $x_S = 0$, of the closed-loop system without disturbances is asymptotically stable.

C2. *Regulation.* For each initial condition $(x_S(0), w(0))$ in a neighbourhood of origin, the solution of the closed-loop system satisfies the condition $\lim_{t\to\infty} e(t) = 0$.

The next theorem states conditions for the existence of a solution to the NRP.

Theorem 1 (Isidori 1995). *The nonlinear regulation problem is locally solvable if and only if the pair* (A, B) *is stabilizable and there exist mappings*

$$x_S = \pi(w) \qquad \text{and} \qquad u = \gamma(w) = \begin{pmatrix} \gamma_1(w) \\ \vdots \\ \gamma_p(w) \end{pmatrix}, \tag{4}$$

with $\pi(0) = 0$ *and* $\gamma(0) = 0$*, both defined in a neighbourhood* $W^\circ \subset W$ *of the origin, satisfying the conditions*

$$\frac{\partial \pi(w)}{\partial w} F_w(w) = F_S(\pi(w), w, \gamma(w)), \tag{5}$$

$$0 = h(\pi(w), w), \tag{6}$$

for all $w \in W^\circ$.

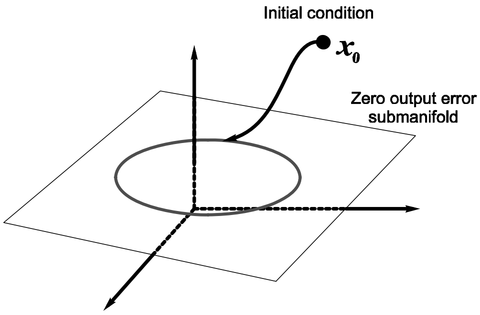

Figure 1. Zero output error submanifold for the tracking problem.

Conditions (5) and (6) are known as the Francis–Isidori–Byrnes equations (FIB) (Byrnes and Isidori 2000) used to find the zero tracking error submanifold. The mapping $x_S = \pi(w)$ represents the steady-state zero output submanifold whose time derivative produces (5), while $u = \gamma(w)$ is the steady-state input which makes invariant this steady- state zero output submanifold.

Figure 1 is a schematic representation of the steady-state zero output submanifold. If the system has an initial condition different from this submanifold, the proposed controller must drive system (1) to the zero output submanifold where the input $u = \gamma(w)$ must be provided in order to make this invariant submanifold. In the following sections FIB equations are used as a tool to address the synchronization phenomena.

3. Synchronization analysis

3.1. Problem statement

Let us consider the following master nonlinear dynamical system:

$$\dot{x}_M = F_M(x_M), \tag{7}$$

$$y_M = h_M(x_M), \tag{8}$$

where x_M is a state vector defined in a neighbourhood W of the origin in $\mathbb{R}^n$ and $F_M(x_M)$ is a smooth vector field. $y_M \in \mathbb{R}^p$ denotes the output of the master system. If system (7)–(8) is chaotic, its trajectories are bounded. Additionally, let us now take a dynamical system

$$\dot{x}_S = F_S(x_S, u), \tag{9}$$

$$y_S = h_S(x_S), \tag{10}$$

where x_S, defined in a neighbourhood U of the origin in $\mathbb{R}^m$, denotes the state vector of the slave system, $u \in \mathbb{R}^p$ is the control command, F_S is a smooth vector field and $y_S \in \mathbb{R}^p$ is the output of the slave system.

In synchronization, under master–slave interconnection, system (7) describes the goal of dynamics, while system (9) represents the experimental system to be controlled. Thus, the *synchronization problem* can be stated as follows. *Given the systems (7) and (9), to determine a signal $u(t)$ which synchronizes the output of the slave system (10) with the output of the master system (8).* That is, given the synchronization error

$$e = h_M(x_M) - h_S(x_S) \tag{11}$$

we find a function $u\,(x_S, x_M)$ such that $\lim_{t\to\infty} e(t) = 0$, for all t and any $x_S(0) \in U$, $x_M(0) \in W$.

3.2. *General nonlinear systems*

Several kinds of synchronization have been defined (Boccaletti *et al* 1999, Femat and Solis-Perales 1999): (i) complete exact synchronization (CES) (where $\|x_S(t) - x_M(t)\| \equiv 0$ for all $t \geqslant 0$), (ii) complete inexact synchronization (where $\|x_S(t) - x_M(t)\| \approx 0$ for all $t \geqslant 0$), (iii) partial synchronization (where at least for one state $x_i(t)$, for any $i \leqslant n$, $\|x_S(t) - x_M(t)\| \neq 0$) and (iv) almost synchronization (where only the phase of the driving system is similar to the response system with a different amplitude).

Actually, the synchronization problem can be addressed as a regulation problem for the above definitions. Since the master dynamical system (7) is similar to system (3) the slave system (9) resembles the system (1) and the synchronization error (11) has similarities with the regulation error (2). Hence, theorem 1 can be adapted to solve the synchronization problem. In order to state the requirements to solve the synchronization problem let us assume that there is a diffeomorphism

$$\underline{x}_S = \Phi_S(x_S) = \begin{pmatrix} \underline{x}_{S1} \\ \underline{x}_{S2} \end{pmatrix}, \tag{12}$$

with $\underline{x}_{S1} \in U_1 \subset \mathbb{R}^{m_1}$, $\underline{x}_{S2} \in U_2 \subset \mathbb{R}^{m_2}$ $(U = U_1 \times U_2)$ and $m_1 + m_2 = m$, such that system (9) becomes

$$\dot{\underline{x}}_{S1} = \underline{F}_{S1}(\underline{x}_{S1}, \underline{x}_{S2}, u), \tag{13}$$

$$\dot{\underline{x}}_{S2} = \underline{F}_{S2}(\underline{x}_{S2}), \tag{14}$$

$$y_S = \underline{h}_S(\underline{x}_{S1}). \tag{15}$$

Let us assume that the first approximation of subsystem (13), characterized by the pair (A, B), is stabilizable where $A = [\partial \underline{F}_{S1}/\partial \underline{x}_{S1}]_{(0,0,0)}$ and $B = [\partial \underline{F}_{S1}/\partial u]_{(0,0,0)}$, whereas the solution of subsystem (14) is Poisson stable. That is system (14) yields trajectories which will return at future time arbitrarily close to any initial condition $x_{S2}(0) \in U_2$ persistently. This fact arises because of the oscillatory nature of chaos.

The following is presented seeking clarity in presentation. Note that when the synchronization error (11) is equal to zero, the states of the slave system related to y_S become a function of the master system states related to y_M, while the states $\underline{x}_{S2}$ evolve independently in a region of U_2 depending on the initial conditions $\underline{x}_{S2}(0)$. In general, when $e = 0$, $\underline{x}_{S1}$ is a function of x_M and $\underline{x}_{S2}$, i.e. there exists a synchronization submanifold $\underline{x}_{S1} = \pi(x_M, \underline{x}_{S2})$ and this submanifold becomes invariant under the input $u = \gamma(x_M, \underline{x}_{S2})$. For instance, if $y_S = \underline{x}_{S1} \in \mathbb{R}$, $\underline{x}_{S2} \in \mathbb{R}^2$ and $x_M \in \mathbb{R}^n$ with $n \geqslant 1$, the projection of the trajectory in the plane $\underline{x}_{S2,1}$–$\underline{x}_{S2,2}$ is fixed by the initial condition $\underline{x}_{S2}(0)$ (see the left-hand side of figure 2), while the altitude (given by $\underline{x}_{S1}$) is determined by the evolution of the master states x_M. In general, when $e = 0$, the hyperplane $h_M(x_M) - \underline{h}_S(\underline{x}_{S1}) = 0$, contains the solution for $\underline{x}_{S2}$ (see the right-hand side of figure 2). This idea is formalized in the next proposition.

Proposition 2. *The synchronization problem is locally solvable if and only if the pair* (A, B) *is stabilizable and there exist mappings*

$$\underline{x}_{S1} = \pi(w), \qquad \textit{and} \qquad u = \gamma(w) = \begin{pmatrix} \gamma_1(w) \\ \vdots \\ \gamma_p(w) \end{pmatrix}, \tag{16}$$

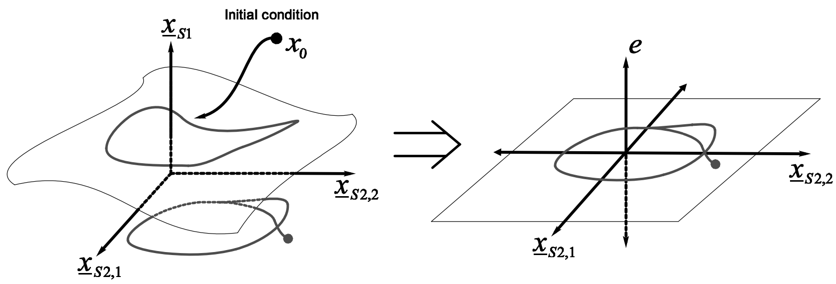

Figure 2. Synchronization submanifold. In original coordinates, the synchronization submanifold is a surface where $\underline{x}_{S1}$ evolves as a function of x_M and $\underline{x}_{S2}$, while the synchronization error remains equal to zero and $\underline{x}_{S2}$ evolves in the plane $e = 0$.

with $\pi(0) = 0$ *and* $\gamma(0) = 0$, *both defined in a neighbourhood* $W^{\circ} \times U_2^{\circ} \subset W \times U_2$ *of the origin, satisfying the conditions*

$$\frac{\partial \pi(w)}{\partial w} F_w(w) = \underline{F}_{S1}\pi(w), \qquad \underline{x}_{S2}, \gamma(w), \tag{17a}$$

$$0 = h_M(x_M) - \underline{h}_S(\pi(w)), \tag{17b}$$

where

$$w = \begin{pmatrix} x_M \\ \underline{x}_{S2} \end{pmatrix}, \qquad \textit{and} \qquad F_w(w) = \begin{pmatrix} F_M(x_M) \\ \underline{F}_{S2}(\underline{x}_{S2}) \end{pmatrix} \tag{18}$$

for all $w \in W^{\circ} \times U_2^{\circ}$.

Proof. Let us define the input

$$u(t) = \alpha(\underline{x}_{S1}, w) = K(\underline{x}_{S1} - \pi(w)) + \gamma(w) \tag{19}$$

such that the matrix $(A + BK)$ is Hurwitz, where $K = [\partial\alpha/\partial\underline{x}_{S1}]_{(0,0)}$. The closed-loop system is

$$\dot{\underline{x}}_{S1} = \underline{F}_{S1}(\underline{x}_{S1}, \underline{x}_{S2}, \alpha(\underline{x}_{S1}, w)), \tag{20a}$$

$$\dot{x}_M = F_M(x_M), \tag{20b}$$

$$\dot{\underline{x}}_{S2} = \underline{F}_{S2}(\underline{x}_{S2}), \tag{20c}$$

and its linear approximation is

$$\dot{\underline{x}}_{S1} = (A + BK)\,\underline{x}_{S1} + Px_M + BLw + \phi(\underline{x}_{S1}, w), \tag{21a}$$

$$\dot{x}_M = S_1 x_M + \psi_1(x_M), \tag{21b}$$

$$\dot{\underline{x}}_{S2} = S_2\underline{x}_{S2} + \psi_2(\underline{x}_{S2}), \tag{21c}$$

where $\phi(\underline{x}_{S1}, w)$, $\psi_1(x_M)$ and $\psi_2(\underline{x}_{S2})$ vanish at the origin with their first-order derivatives, and A, B and K are defined previously, while $P = [\partial\underline{F}_{S1}/\partial\underline{x}_{S2}]_{(0,0,0)}$, $L = [\partial\alpha/\partial w]_{(0,0)}$, $S_1 = [\partial F_M/\partial x_M]_{(0)}$ and $S_2 = [\partial\underline{F}_{S2}/\partial\underline{x}_{S2}]_{(0)}$. Hence, the Jacobian matrix of the closed-loop system at the equilibrium $(\underline{x}_{S1}, w) = (0, 0)$ has the following form:

$$\begin{pmatrix} (A + BK) & \star \\ 0 & S \end{pmatrix}, \tag{22}$$

where $S = \mathrm{diag}(S_1, S_2)$. Since hypotheses (7) and (14) are Poisson stable, matrix S must have all its eigenvalues on the imaginary axis. Thus, using the central manifold theory, we deduce the existence of a central manifold at $(\underline{x}_{S1}, w) = (0, 0)$. This manifold can be expressed as the graph of a mapping $\underline{x}_{S1} = \pi(w)$, with $\pi(w)$ satisfying equation (17*a*).

Let us consider a real number $R > 0$, and let w° be a point in $W^\circ \times U_2^\circ$, with $\|w^\circ\| < R$, since by hypothesis of neutral stability, the equilibrium $w = 0$ is stable, it is possible to choose R so that the solution $w(t)$ of (20*b*)–(20*c*) satisfying $w(0) = w^\circ$ remains in $W^\circ \times U_2^\circ$ for all $t \geqslant 0$. If $\underline{x}_{S1}(0) = \pi(w^\circ)$, the corresponding solution $x(t)$ of (20*a*) will be such that $x(t) = \pi(w(t))$ for all $t \geqslant 0$ because the manifold $x = \pi(w)$ is by definition invariant under the flow of (20*a*). The rest of the proof is similar to that presented in Isidori (1995) for the regulation problem. □

In regard to proposition 2, the following remarks are in order: (i) the feedback interconnection was selected as a static input and considers full information (state feedback controller); however, further schemes have been proposed to solve the regulation problem which can be extended to the synchronization problem, including the dynamic controller for the output feedback (Isidori 1995), robust output feedback controllers (García-Sandoval *et al* 2007), discrete controllers (Castillo-Toledo and Di'Gennaro 2002), among others. (ii) The input 19 stands for a diffusive coupling as K induces all eigenvalues of $(A + BK)$ to lie on the open-left side of a complex plane.

Also note that if $\dim(\underline{x}_{S2}) = 0$, in the framework of the generalized synchronization, the slave system is totally synchronized; i.e. there exists the map $x_S = \Phi_S^{-1}(\pi(x_M))$ which allows us to calculate x_S from x_M, despite a different dimension between the master and the slave, by means of the mapping $\pi(x_M)$ which is a contraction if $m_1 < n$ or an immersion if $m_1 > n$. This relationship is schematically presented in the interconnections diagram of figure 3(*a*). Also note that, if in addition $m_1 = n$, because of the map $\Phi_S^{-1}(\pi(\cdot))$, not necessarily $x_S(t) = x_M(t)$, unless both the master and slave systems were identical, since in this case, the map $\Phi_S^{-1}(\pi(\cdot))$ would be the identity and the traditional complete exact synchronization would be reached. On the other hand, if $\dim(\underline{x}_{S2}) \neq 0$, only the first m_1 states of $\underline{x}_S$ are synchronized with the combined variables x_M and $\underline{x}_{S2}$, i.e. when the synchronization is achieved (and therefore $e = 0$), the m_1 states of the vector $\underline{x}_{S1}$ reside in the invariant submanifold $\pi(w)$ which depends in general on $m_2 + n$ states. Hence, in some sense, m_1 states are synchronized with the master system; however, they may also depend on the remaining m_2 states of the slave system, $\underline{x}_{S2}$, therefore, one gets the map $\Phi_{S1}(x_S) = \pi(x_M, \Phi_{S2}(x_S))$ (see figure 3(*b*)). In this case only partial synchronization will be reached. Note that if there exists a diffeomorphism

$$\underline{x}_M = \Phi_M(x_M) = \begin{pmatrix} \underline{x}_{M1} \\ \underline{x}_{M2} \end{pmatrix} \tag{23}$$

such that (7) and (8) become

$$\dot{\underline{x}}_{M1} = \underline{F}_{M1}(\underline{x}_{M1}), \tag{24a}$$

$$\dot{\underline{x}}_{M2} = \underline{F}_{M2}(\underline{x}_{M1}, \underline{x}_{M2}), \tag{24b}$$

$$y_M = \underline{h}_M(\underline{x}_{M1}), \tag{24c}$$

with $\underline{x}_{M1} \in W_1 \subset \mathbb{R}^{n_1}$, $\underline{x}_{M2} \in W_2 \subset \mathbb{R}^{n_2}$ $(W = W_1 \times W_2)$ and $n_1 + n_2 = n$, then the invariant synchronization submanifold, $\pi(w)$, depends only on $m_2 + n_1$ states and in some sense the synchronization is achieved for the m_1 states of the vector $\underline{x}_{S1}$ with the n_1 states of $\underline{x}_{M1}$, as well as the m_2 independent states of the slave system $\underline{x}_{S2}$. Since the states $\underline{x}_{M2}$ are not involved in the synchronization, in the following we only consider the dynamics of the observable part of

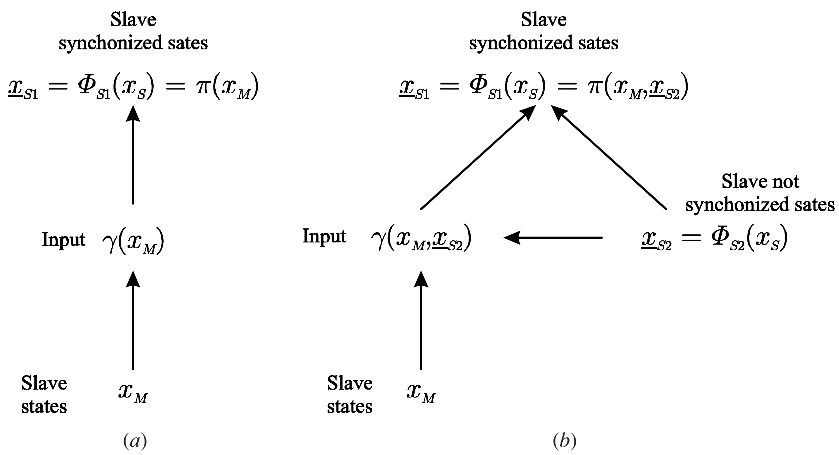

Figure 3. Interconnections diagram for generalized synchronization. (*a*) Total synchronization, when $\dim(\underline{x}_{S2}) = 0$. (*b*) Partial synchronization, when $\dim(\underline{x}_{S2}) \neq 0$.

x_M, i.e. $\underline{x}_{M1}$. The existence of diffeomorphisms (12) and (23) depends on local reachability and local observability, respectively, and can be obtained following the procedures described in Femat and Solis-Perales (2008).

3.3. Details on designing synchronization force

The synchronization equations (17) are valid for multi-input multi-output systems, since $u \in \mathbb{R}^p$ and $e \in \mathbb{R}^p$; however, in this section a thorough analysis for single-input single-output affine systems is conducted. Let us consider the dynamical system

$$\dot{x}_M = f_M(x_M), \tag{25}$$

$$y_M = h_M(x_M), \tag{26}$$

where $x_M \in \mathbb{R}^n$ represents the states of the master system defined in a neighbourhood W of the origin in $\mathbb{R}^n$ and f_M is a smooth vector field. $y_M \in \mathbb{R}$ is the output of the master system. Additionally, let us define the dynamics of the slave system as

$$\dot{x}_S = f_S(x_S) + g_S(x_S)u, \tag{27}$$

where $x_S \in \mathbb{R}^m$ defined in a neighbourhood U of the origin in $\mathbb{R}^m$ denotes the state vector, $u \in \mathbb{R}$ is the control command, $f_S(x_S)$ and $g_S(x_S)$ are the smooth vector fields and $y_S \in \mathbb{R}$ is the output of the slave system. The synchronization error is defined as $e = h_M(x_M) - h_S(x_S)$, where $e \in \mathbb{R}$. Furthermore, it is worth mentioning that this analysis may be applied to the different vector fields f_M and f_S and even in the case where the slave and master systems have different dimensions. The next assumption is instrumental to the following analysis.

Assumption 3. *Let us consider that the relative degree of (27) is well defined and equal to ρ.*

Since the relative degree is well defined, it is possible to find a diffeomorphism which transforms system (27) into a normal form. Moreover, for the synchronization analysis, one may split the inner dynamics in order to consider the asymptotic stable and Poisson stable modes; hence one may assume that there exists a diffeomorphism

$$\underline{x}_S = \Phi_S(x_S) = \begin{pmatrix} \zeta \\ \eta \\ \underline{x}_{S2} \end{pmatrix}, \tag{28}$$

where $\zeta_i = L_{f_S}^{i-1} h_S(x_S), i = 1, 2, \ldots, \rho$, with $L_{f_S}^{0} h_S(x_S) = h_S(x_S)$, while $\eta_j = \phi_{S,j}(x_S)$, $j = 1, 2, \ldots, m_1 - \rho$, with $\phi_{S,j}(x_S)$ such that $L_{g_S}\phi_{S,j}(x_S) = 0$, which transforms system (27) into the normal form

$$\dot{\zeta}_i = \zeta_{i+1}, \qquad i = 1, 2, \ldots, \rho - 1, \tag{29a}$$

$$\dot{\zeta}_\rho = a(\zeta, \eta, \underline{x}_{S2}) + b(\zeta, \eta, \underline{x}_{S2})u, \tag{29b}$$

$$\dot{\eta} = q(\zeta, \eta, \underline{x}_{S2}), \tag{29c}$$

$$\dot{\underline{x}}_{S2} = \underline{F}_{S2}(\underline{x}_{S2}), \tag{29d}$$

where $a(\zeta, \eta, \underline{x}_{S2}) = \left[L_{f_s}^{\rho} h_S(x_S)\right]_{x_S=\Phi_S^{-1}(\underline{x}_S)}$, $b(\zeta, \eta, \underline{x}_{S2}) = \left[L_{g_S} L_{f_s}^{\rho-1} h_S(x_S)\right]_{x_S=\Phi_S^{-1}(\underline{x}_S)}$ and $q_j(\zeta, \eta, \underline{x}_{S2}) = \left[L_{f_s}\phi_{S,j}(x_S)\right]_{x_S=\Phi_S^{-1}(\underline{x}_S)}$ for $j = 1, 2, \ldots, m_1 - \rho$. Here, we are assuming that equation (29*d*) is the Poisson stable, while $\dot{\eta} = q\,(0, \eta, 0)$ is asymptotically stable.

Note that equation (29*d*) is identical to equation (14) while equations (29*a*)–(29*c*) represent subsystem (13). On the other hand, when $\zeta = 0$, subsystems (29*c*) and (29*d*) form a central manifold around $\eta = 0$.

Since the synchronization equations (17) hold for any dimension for the slave and master systems, depending on the master system dimension and the relative degree of system (27), two cases are considered.

Case 1. If $\rho < n$, the diffeomorphism $\xi = \Phi_M(x_M)$, where $\xi_i = L_{f_M}^{i-1} h_M(x_M), i = 1, 2, \ldots, \rho$, while $\xi_j = \phi_{M,j}(x_M)$, $j = \rho + 1, \ldots, n$, transforms (25) into

$$\begin{aligned} \dot{\xi}_i &= \xi_{i+1}, \qquad i = 1, 2, \ldots, \rho - 1, \\ \dot{\xi}_\rho &= r_1(\xi), \\ \dot{\xi}_j &= r_{j-\rho+1}(\xi), \qquad j = \rho + 1, \ldots, n, \end{aligned} \tag{30a}$$

where $r_j(\xi) = \left[L_{f_M}\phi_{M,j}(x_M)\right]_{x_M=\Phi_M^{-1}(\xi)}$ for $j = 1, \ldots, n - \rho + 1$.

Case 2. If $\rho \geqslant n$, the diffeomorphism $\xi = \Phi_M(x_M) = \mathrm{col}\left\{L_{f_M}^{i-1} h_M(x_M), i = 1, 2, \ldots, n\right\}$, transforms (25) into

$$\begin{aligned} \dot{\xi}_i &= \xi_{i+1}, \qquad i = 1, 2, \ldots, n - 1, \\ \dot{\xi}_n &= r(\xi), \end{aligned} \tag{31}$$

where $r(\xi) = \left[L_{f_M}^{n} h_M(x_M)\right]_{x_M=\Phi_M^{-1}(w)}$.

Note that for case 2 the master system dimension is necessarily smaller than the slave system dimension, while for case 1 such dimension may be smaller, greater or equal.

Using the normal form for both the master and slave systems (equations (31) or (30*a*) and (29*a*)), the synchronization error can be written as

$$e = \xi_1 - \zeta_1. \tag{32}$$

Now let us analyse each case.

3.3.1. Case 1: $\rho < n$. Considering (30*a*) and (29) for the master and slave systems, respectively, and (32) as the synchronization error, from equations (17) we deduce the following.

The states ζ are completely synchronized with the first ρ states of ξ , since

$$\zeta_i = \pi_i(\xi) = \xi_i, \qquad i = 1, 2, \ldots, \rho, \tag{33}$$

which in original coordinates is equivalent to

$$L_{f_S}^{i-1} h_S(x_S) = L_{f_M}^{i-1} h_M(x_M), \qquad i = 1, 2, \ldots, \rho. \tag{34}$$

From (34), we deduce that the hyperplane defined by $L_{f_S}^{i-1} h_S(x_S) - L_{f_M}^{i-1} h_M(x_M) = 0$ contains the evolution of the remaining states of $\underline{x}_S$, i.e. η and $\underline{x}_{S2}$. The synchronization input necessary to obtain (34) is given by the mapping

$$\gamma = \frac{r_1(\xi) - a(\xi, \eta_{ss}, \underline{x}_{S2})}{b(\xi, \eta_{ss}, \underline{x}_{S2})}, \tag{35}$$

where η_{ss} is the solution of

$$\dot{\eta}_{ss} = q(\xi, \eta_{ss}, \underline{x}_{S2}), \tag{36}$$

for a given initial condition $\eta_{ss}(0) = \eta_0$. However, since $\dot{\eta} = q(0, \eta, 0)$ is asymptotically stable, there exists a class $\mathcal{K}$ function α_w and a class $\mathcal{KL}$ function β such that

$$\|\eta_{ss}(t)\| \leqslant \beta(\eta_0, t) + \alpha_w(w), \tag{37}$$

where $w = \left(\xi^T \;\; \underline{x}_{S2}^T\right)^T$. Note that w contains the master and slave variables. Given a large enough time $t \geqslant T$, $\beta(\eta_0, t) \to 0$, hence for $t \geqslant T$, η_{ss} does not depend any longer on the initial condition. Then, η_{ss} together with

$$\dot{w} = R(w), \tag{38}$$

where $R(w) = \text{col}\{\xi_2, \ldots, \xi_\rho, r_1(\xi), \ldots, r_{n-\rho+1}(\xi), \underline{F}_{S2}(\underline{x}_{S2})\}$, generate a central manifold and for this reason for large enough time $\eta_{i,ss} = \pi_{i+\rho}(w), i = 1, 2, \ldots, m_1 - \rho$ and γ eventually depend only on w. From the previous discussion, one may conclude that when synchronization is achieved, ζ is totally synchronized, while η is, in some sense, synchronized with the master system; however, it also depends on the Poisson stable states of the slave system $\underline{x}_{S2}$. Note that if $\dim(\underline{x}_{S2}) = 0$, the slave system is totally synchronized, i.e. there exists the map $x_S = \Phi_S^{-1}(\pi(\Phi_M(x_M)))$. Here map π allows that $n \neq m$.

Another interesting case is when the master and slave systems are identical, then the same diffeomorphism can be defined for both the systems, i.e. $\Phi_S(x_S) = \Phi_M(x_M)$. However, if the initial conditions for the Poisson stable subsystems are not the same, their time evolution may differ and only the partial synchronization may be achieved. On the other hand, if the initial conditions of the Poisson stable subsystems are identical, the total synchronization can be achieved.

3.3.2. Case 2: $\rho \geqslant n$. Considering (31) and (29) for the master and slave systems, respectively, and (32) as the synchronization error, from equations (17) we deduce the following.

The first n states of ζ are directly synchronized with ξ, since

$$\zeta_i = \pi_i(\xi) = \xi_i, \qquad i = 1, 2, \ldots, n, \tag{39}$$

while the $\rho - n$ following states of ζ are in the tangent space of ξ_n since

$$\zeta_i = \pi_i(\xi) = L_R^{i-n-1} r(\xi), \qquad i = n+1, n+2, \ldots, \rho, \tag{40}$$

where $R(\xi) = \mathrm{col}\{\xi_2, \ldots, \xi_n, r(\xi)\}$. In original coordinates

$$L_{f_S}^{i-1} h_S(x_S) = L_{f_M}^{i-1} h_M(x_M), \qquad i = 1, 2, \ldots, n, \tag{41}$$

$$L_{f_S}^{i-1} h_S(x_S) = L_R^{i-n-1} L_{f_M}^{n} h_M(x_M), \qquad i = n+1, n+2, \ldots, \rho, \tag{42}$$

with the elements of R as $R_i(x_M) = L_{f_M}^{i} h_M(x_M), i = 1, 2, \ldots, n$. Therefore, the complete synchronization is achieved for the first m_1 state of $\underline{x}_S$. On the other hand, the synchronization input is given by the mapping

$$\gamma = \frac{r(\xi) - a(\xi, \eta_{ss}, \underline{x}_{S2})}{b(\xi, \eta_{ss}, \underline{x}_{S2})}, \tag{43}$$

where η_{ss} is the solution of

$$\dot{\eta}_{ss} = q(\xi, \eta_{ss}, \underline{x}_{S2}), \tag{44}$$

which, for the same reasons described in the case 1, forms a central manifold with ξ and $\underline{x}_{S2}$ and therefore the synchronization input γ depends only on ξ and $\underline{x}_{S2}$.

Finally, for both the cases, if $\dim(\underline{x}_{S2}) > 0$ but $\dim(\eta) = 0$, only ζ is synchronized with x_M and $\underline{x}_{S2}$ only affect the synchronization input γ.

4. Examples

In this section we present three workbench examples.

Example 1. Identical master and slave systems.

Consider a Duffing system

$$y''(t) - y(t) + y^3(t) + \delta\dot{y}(t) = \tau(t), \tag{45}$$

where $\tau(t) = \alpha \sin(ct + \theta)$ represents an oscillatory driving signal, which can be described by the dynamical equation $\tau''(t) = -c^2\tau(t)$. This system is used as a master system to synchronize the system

$$z''(t) - z(t) + z^3(t) + \delta\dot{z}(t) = \widehat{\tau}(t) + u(t), \tag{46}$$

where $u(t)$ is the input used for the synchronization, while $\widehat{\tau}(t)$ is an oscillatory driven input. We consider that the synchronization error $e = z - y$. The state space representation of the master and slave systems is

master: $\dot{x}_{M,1} = x_{M,2}, \quad \dot{x}_{M,2} = F(x_M), \quad \dot{x}_{M,3} = cx_{M,4}$ and $\dot{x}_{M,4} = -cx_{M,3}$,

slave: $\dot{x}_{S,1} = x_{S,2}, \quad \dot{x}_{S,2} = F(x_S) + u(t), \quad \dot{x}_{S,3} = cx_{S,4}$ and $\dot{x}_{S,4} = -cx_{S,3}$,

where $F(x) = x_1 - x_1^3 - \delta x_2 + x_3$, while the synchronization error $e = x_{S,1} - x_{M,1}$. These systems can be written as (30*a*) and (29) if we define $\xi = x_M, \zeta = (x_{S,1} \;\; x_{S,2})^T, \underline{x}_{S2} = (x_{S,3} \;\; x_{S,4})^T, a(\zeta, \underline{x}_{S2}) = \zeta_1 - \zeta_1^3 - \delta\zeta_2 + \underline{x}_{S2,1}$ and $b(\zeta, \underline{x}_{S2}) = 1$. Note that $\dim(\eta) = 0$, since both $\underline{x}_{S2}$ and the last two states of ξ are Poisson stable, when synchronization is achieved $x_{S,1} = x_{M,1}$ and $x_{S,2} = x_{M,2}$; however, if the initial conditions of the last two states of the master and slave systems are different, then $x_{S,3} \neq x_{M,3}$ and $x_{S,4} \neq x_{M,4}$. In this case, the partial synchronization is achieved. On the other hand, if the initial conditions of the last two states of the master and slave systems are identical, then $x_S = x_M$ and the complete synchronization is obtained. Figure 4 shows the error $e = x_S - x_M$ obtained by using a linearizing feedback controller

$$u(t) = F(x_M) - F(x_S) + k_1(\xi_1 - \zeta_1) + k_2(\xi_2 - \zeta_2), \tag{47}$$

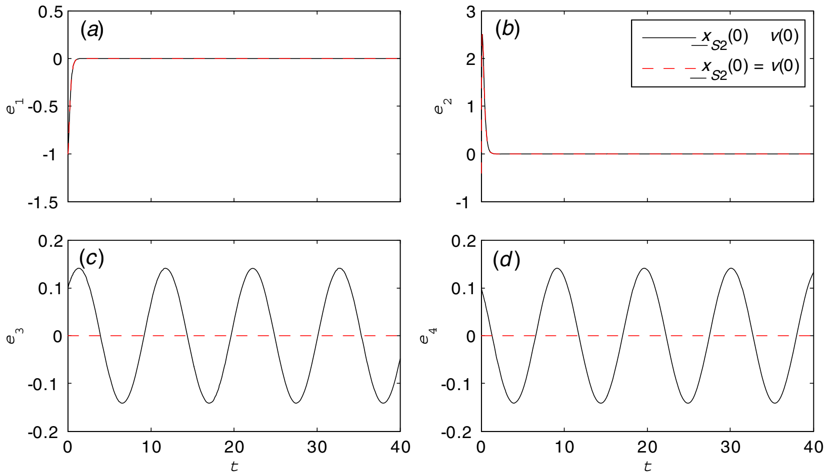

Figure 4. Synchronization error for example 1. Solid line: different initial conditions for $\underline{x}_{S2}$. Dashed line:same initial conditions for $\underline{x}_{S2}$.

for both the cases. The parameters used for the simulation are $\delta = 0.2, c = 0.6, k_1 = 50$ and $k_2 = 15$, while the initial conditions are $x_M(0) = (0.5 \;\; 0.1 \;\; 0.5 \;\; 0.0)^T, x_S(0) = (1.5 \;\; 0.5 \;\; 0.4 \;\; -0.1)^T$ and $x_S(0) = (1.5 \;\; 0.5 \;\; 0.5 \;\; 0)^T$, respectively.

Example 2. Synchronization of different systems. Let us consider now the Rössler model as a master system and the Lorenz model as its slave system, where mappings f_M, h_M, f_S, g_S and h_S are respectively: Rössler's:

$$f_M(x_M) = \begin{pmatrix} -x_{M,2} - x_{M,3} \\ x_{M,1} + a x_{M,2} \\ a + x_{M,3}(x_{M,1} - b) \end{pmatrix}, \qquad h_M(x_M) = x_{M,2}, \tag{48}$$

Lorenz's:

$$f_S(x_S) = \begin{pmatrix} \sigma(x_{S,2} - x_{S,1}) \\ \rho x_{S,1} - x_{S,2} - x_{S,1}x_{S,3} \\ -\beta x_{S,3} + x_{S,1}x_{S,2} \end{pmatrix}, \qquad g_S(x_S) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad h_S(x_S) = x_{S,1}, \tag{49}$$

where a, b, σ, ρ and β are the positive constants. The relative degree of the slave system is 2. Therefore, by defining $\zeta = (x_{S,1} \;\; \sigma(x_{S,2} - x_{S,1}))^T$ and $\eta = x_{S,3}$, the normal form of (49) is similar to (29*a*)–(29*d*) where $a(\zeta, \eta) = \sigma(\rho - 1)\zeta_1 - (1 + \sigma)\zeta_2 - \sigma\zeta_1\eta, b(\zeta, \eta) = \sigma$ and $q(\zeta, \eta) = -\beta\eta + \zeta_1^2 + \sigma^{-1}\zeta_1\zeta_2$. Note that $\dim(\underline{x}_{S2}) = 0$ and that $q(0, \eta) = -\beta\eta$ is asymptotically stable. On the other hand, if one defines $\xi = (x_{M,2} \;\; x_{M,1} + a x_{M,2} \;\; x_{M,3} - a/b)^T$, then the Rössler model is similar to (30*a*) with $r_1(\xi) = -\xi_1 + a(\xi_2 + 1/b) - \xi_3$ and $r_2(\xi) = -b\xi_3 + (\xi_3 + a/b)(\xi_2 - a\xi_1)$. Note that when $\xi = 0, r_2(\xi) = -b\xi_3$, which is asymptotically stable. The input γ given by (35) is

$$\gamma(\xi) = \left(1 - \rho - \frac{1}{\sigma}\right)\xi_1 + \left(1 + \frac{1 + a}{\sigma}\right)\xi_2 - \frac{1}{\sigma}\xi_3 + \xi_1\eta_{ss}, \tag{50}$$

where η_{ss} is the solution of $\dot{\eta}_{ss} = -\beta\eta_{ss} + \zeta_1^2 + \sigma^{-1}\zeta_1\zeta_2$ given by

$$\eta_{ss}(t) = \eta_0 \mathrm{e}^{-\beta t} + \int_0^t \xi_1(\tau)\left[\xi_1(\tau) + \frac{1}{\sigma}\xi_2(\tau)\right]\mathrm{e}^{\beta(t-\tau)}\mathrm{d}\tau. \tag{51}$$

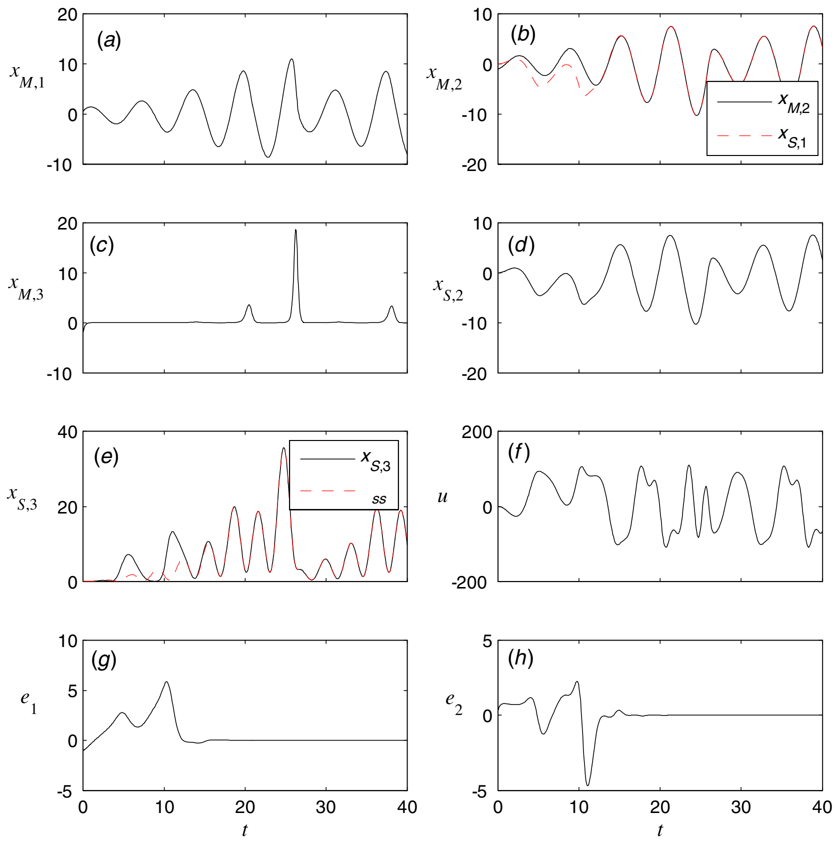

Figure 5. Lorenz system and Rössler system synchronization for example 2. (*a*) $x_{M,1}$, (*b*) synchronization output and reference, $x_{S,1}$ and $x_{M,2}$, (*c*) $x_{M,3}$, (*d*) $x_{S,2}$, (*e*) inner dynamic, $x_{S,3}$ and its steady state, η_{ss}, (*f*) input, (*g*) and (*h*) synchronization errors.

For large enough time, the first term of (51) disappears and η_{ss} depends only on ξ_1 and ξ_2. Therefore, when synchronization is achieved, $x_{S,1} = x_{M,2}$ and $x_{S,2} = x_{M,1}/\sigma + (1 + a/\sigma)\, x_{M,2}$, while $x_{S,3} = \int_0^t x_{M,2}(\tau)\, x_{S,2}(\tau)\, \mathrm{e}^{\beta(t-\tau)} \mathrm{d}\tau$ and the complete synchronization is achieved. Considering that full information is available, we use the input

$$u = k_1(\zeta_1 - \xi_1) + k_2(\zeta_2 - \xi_2) + \widehat{\gamma}(\xi, \eta), \tag{52}$$

where

$$\widehat{\gamma}(\xi, \eta) = \left(1 - \rho - \frac{1}{\sigma}\right)\xi_1 + \left(1 + \frac{1 + a}{\sigma}\right)\xi_2 - \frac{1}{\sigma}\xi_3 + \xi_1\eta. \tag{53}$$

Figure 5 depicts the closed-loop behaviour when this controller is used. The parameters used are: $\sigma = 10$, $\rho = 28$, $\beta = 8/5$, $a = 0.2$, $b = 5.7$, $k_1 = -20$ and $k_2 = -10$, while the initial conditions are: $x_M(0) = (0.5\ \ -1\ \ 2)^T$ and $x_S(0) = (0\ \ 0\ \ 0)^T$.

Example 3. Synchronization of systems with different order. Now we consider the synchronization of the Duffing equation similar to the one considered in example 1,

$$y''(t) - y(t) + y^3(t) + \delta\dot{y}(t) = \tau(t) + u(t), \tag{54}$$

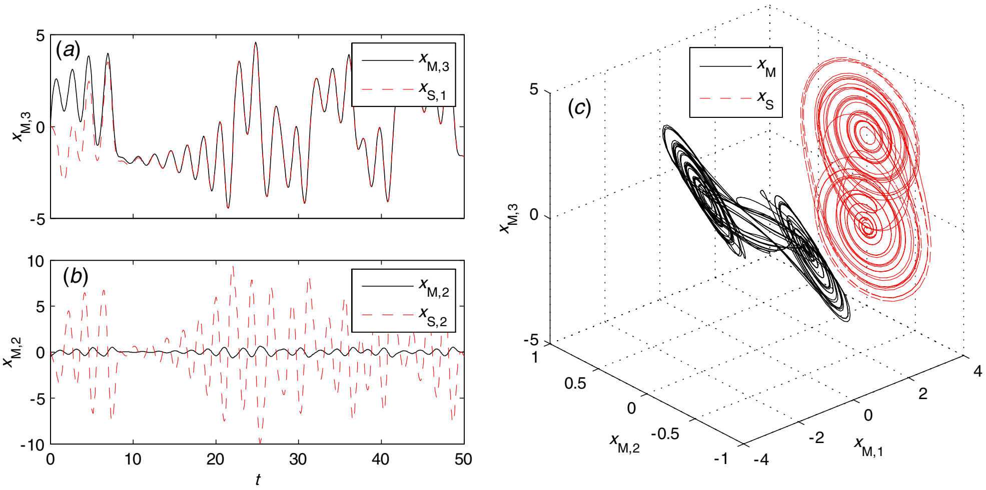

Figure 6. Synchronization of example 3. (*a*) Synchronization output and reference (*b*) $x_{M,2}$ and $x_{S,2}$, with the same frequency but different amplitude. (*c*) Chua phase portrait and its projection over plane $(x_{M,2}, x_{M,3})$.

where $\tau(t) = \alpha \sin(ct + \theta)$ is a driving signal, with the Chua system (an electronic circuit with a nonlinear resistive element) being the master system. The circuit equations can be written as a third-order system which is given by the following dimensionless form:

$$\dot{x}_{M,1} = \gamma_1(x_{M,2} - x_{M,1} - f(x_{M,1})), \tag{55}$$

$$\dot{x}_{M,2} = x_{M,1} - x_{M,2} + x_{M,3}, \tag{56}$$

$$\dot{x}_{M,3} = -\gamma_2 x_{M,2}, \tag{57}$$

where $f(x_{M,1}) = \gamma_3 x_{M,1} + 0.5(\gamma_4 - \gamma_3)[|x_1 + 1| - |x_1 - 1|]$, while the slave system is $\dot{x}_{S,1} = x_{S,2}, \dot{x}_{S,2} = F(x_S) + u(t), \dot{x}_{S,3} = cx_{S,4}$ and $\dot{x}_{S,4} = -cx_{S,3}$, with $F(x_S) = x_{S,1} - x_{S,1}^3 - \delta x_{S,2} + x_{S,3}$. Note that $x_S \in \mathbb{R}^4$ and $x_M \in \mathbb{R}^3$, hence the slave system has higher dimension than the master system. Defining $\zeta = (x_{S,1} \;\; x_{S,2})^T, \underline{x}_{S2} = (x_{S,3} \;\; x_{S,4})^T, a(\zeta, \underline{x}_{S2}) = \zeta_1 - \zeta_1^3 - \delta\zeta_2 + \underline{x}_{S2,1}$ and $b(\zeta, \underline{x}_{S2}) = 1$, the Duffing system can be written as (29). On the other hand, the Chua system can be written as (30*a*), if we define $\xi = (x_{M,3} \;\; -\gamma_2 x_{M,2} \;\; -\gamma_2 x_{M,1})^T . r_1(\xi) = -\xi_2 - \gamma_2\xi_1 + \xi_3$ and $r_2(\xi) = \gamma_1[\xi_2 - \xi_3 - \widehat{f}(\xi_3)]$, where $\widehat{f}(\xi_3) = \gamma_3\xi_3 + 0.5(\gamma_4 - \gamma_3)(|\xi_3 + \gamma_2| - |\xi_3 - \gamma_2|)$. When synchronization is achieved $x_{S,1} = x_{M,3}$ and $x_{S,2} = -\gamma_2 x_{M,2}$, this explains the Chiral behaviour (since $\text{sign}(x_{S,2}) = -\text{sign}(x_{M,2})$) (Femat and Solis-Perales 2008) and why $x_{S,2}$ and $x_{M,2}$ have the same oscillatory frequencies but with a different amplitude (its relationship is given by $-\gamma_2$). $x_{S,3}$ and $x_{S,4}$ remain independent and the synchronization input (35) must be $\gamma(\xi, \underline{x}_{S2}) = -\xi_2 - \gamma_2\xi_1 + \xi_3 - \zeta_1 + \zeta_1^3 + \delta\zeta_2 - \underline{x}_{S2,1}$, which depends on ξ and $\underline{x}_{S2}$.

Figure 6 shows the synchronization behaviour using the input $u = \gamma(\xi, \underline{x}_{S2}) + k_1(\xi_1 - \zeta_1) + k_2(\xi_2 - \zeta_2)$. As expected, after a transient period synchronization is achieved and $x_{S,1} = x_{M,3}$, while $x_{S,2} = -\gamma_2 x_{M,2}$ (see figures 6(*a*) and 6(*b*)). The Chua phase portrait was plotted in figure 6, while $(-x_{S,2}/\gamma_2, x_{S,1})$ was plotted in the plane $(x_{M,2}, x_{M,3})$, which is exactly the projection of Chua's system states.

5. Conclusions

A synchronization analysis of the chaotic systems has been derived as an extension of a classical problem in control theory: the *regulation problem*. As a result, the synchronization manifold and the driving force for synchronization can be established by solving a set of partial differential equations. Using this approach, we can explain phenomena observed in the chaotic synchronization; for instance, the complete and partial-state synchronization as well as the phase synchronization. This methodology can be systematically applied to predict the conditions for the complete or partial-state synchronization and, through the synchronization manifold, to elucidate the relation between the master and slave states. An advantage of this approach lies in the possibility of analysing the synchronization phenomena for strictly different systems as well as for different-order systems. Finally, some workbench examples have been presented to illustrate our results.

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